Last time:

A set A is denumerable if

equivalent conditions

- There is a bijective function $f: \mathbb{N} \to A$
- The elements of A can be listed
$$A = \{a_1, a_2, a_3, \dots\}$$
and it's infinite.

Def: If a set A is denumerable, we say it has size $\aleph_0$ (aleph-naught)
$$|A| = \aleph_0$$

Exercises:
Prove there is a bijection

(1) • $\mathbb{N} \longrightarrow \mathbb{Z} - \{2\}$

(2) • $\mathbb{N} \longrightarrow \{n \in \mathbb{Z} : n$ is either even or is a multiple of $3\}$.
or is both.

(1) $\mathbb{Z} - \{2\} = \{\ldots, -2, -1, 0, 1, 3, 4, 5, \ldots\}$

$= \{0, 1, -1, \cancel{2}, -2, 3, -3, \ldots\}$

$= \{0, 1, -1, -2, 3, -3, \ldots\}$

Prop: If A is any denumerable set, and $B \subseteq A$, then either B is finite or B is denumerable.

Pf Sketch: Let's do this for the special case $A = \mathbb{N}$.

Then $B \subseteq \mathbb{N}$. I can list the elements of B

$B = \{b_1, b_2, b_3, b_4, \ldots\}$

where b_1 = smallest element of B

b_{n+1} = smallest element of $B - \{b_1, b_2, \ldots, b_n\}$. ∎

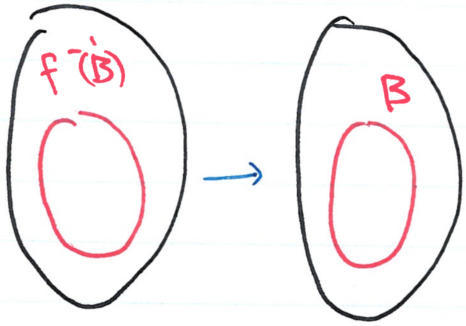
$0, 1, -1, \cancel{2}, -2, 3, \ldots$

$\leftarrow$

$0, 1, -1, -2, 3, \ldots$

Comment: If A is any denumerable set, then

$f: \mathbb{N} \longrightarrow A$



Questions like "How big is the subset B"

are equivalent to "How big is $f^{-1}(B)$"

because f is bijective.

Comment: From the standpoint of adding numbers, multiplying etc. these "reordering" functions throw away all that information. When thinking about sizes of sets, this is ok.

Sizes of Sets.

Denumerable Sets.

$$0 < 1 < 2 < 3 < \ldots < \aleph_0 < ?$$

Proposition: Suppose A and B are denumerable sets

$$A = \{a_1, a_2, a_3, \dots\}$$
$$B = \{b_1, b_2, b_3, \dots\}$$

Then, • (1) $A \cap B$ is either denumerable or finite

• (2) $A \cup B$ is denumerable.

Exercise

Proof: Note that $A \cap B$ is a subset of A, so it automatically follows that $A \cap B$ is either denumerable.

Still, we'll prove it.

(1) • Take the list $(a_1, a_2, a_3, \dots)$ and remove every element that isn't in B.

$\checkmark$ = it is in $A \cap B$

1	$a_1 \checkmark$		a_1
2	$\cancel{a_2}$ ✗	$\rightsquigarrow$	a_3
3	$a_3 \checkmark$		a_4
4	$a_4 \checkmark$		a_6
5	$\cancel{a_5}$ ✗		$\vdots$
	$\vdots$		

(Will write the rigorous proof using WOP).

(2) $A \cup B$ is $\{a_1, b_1, a_2, b_2, \ldots\}$

(of course, throwing away any repeats we see.)

"$\aleph_0 + \aleph_0 = \aleph_0$"

<u>Prop</u>: $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Or in other words,

if A & B are denumerable, then $A \times B$ is denumerable.

$\mathbb{N}$	1	2	3	4	5
1	1	2	4	7	11
2	3	5	8	12	
3	6	9	13		
4	10	14			
⋮	15				

Proof next time.

More rigorous proofs:

Prop: Let A be denumerable, and $B \subseteq A$. Then B is either denumerable or finite

Pf: If B is finite, we are done. Assume B is infinite.

Pick a bijection $f: \mathbb{N} \to A$. Define $c_1, c_2, c_3, \ldots \in \mathbb{N}$ by

$\forall n \in \mathbb{N}$, $c_n =$ the smallest elt of $f^{-1}(B) - \{c_1, c_2, \ldots, c_{n-1}\}$

Since $f^{-1}(B)$ is infinite, the set $f^{-1}(B) - \{c_1, \ldots, c_{n-1}\}$ is nonempty, so by WOP, the numbers c_n are well-defined.

Let $g: \{c_1, c_2, c_3, \ldots\} \to B$, $g(c_i) = f(c_i)$ be the restriction of f to the domain $\{c_1, c_2, \ldots\}$.

I claim g is bijective. Inj: If $g(c_i) = g(c_j)$, then $f(c_i) = f(c_j)$ $\Rightarrow c_i = c_j$, since f is inj.

Surj: ~~Suppose~~ Pick any $b \in B$. Then since f is surj., $\exists c \in \mathbb{N}$, $f(c) = b$. Then $c \in f^{-1}(B) = \{c_1, c_2, \ldots\}$ so $\exists i \in \mathbb{N}$, $c = c_i$. Then $g(c_i) = f(c_i) = b$. ∎

Example 1: Since $\mathbb{Z} - \{2\} \subseteq \mathbb{Z}$, and $\mathbb{Z}$ is denumerable, it immediately follows that $\mathbb{Z} - \{2\}$ is denumerable.

Prop: If A and B are denumerable sets, then $A \cap B$ is denumerable.

Pf: Repeat the same proof as above, but using "$A \cap B$" in place of B.

Prop: Let A be a denumerable set. Suppose $f: A \to B$ is a surjection. Then B is either finite or denumerable.

Pf: If B is finite, we are done. Assume B is infinite.

Let $g: \mathbb{N} \to A$ be a bijection. Define natural numbers $c_1, c_2, c_3, \dots$ by

$c_n =$ the smallest element of the set

$$\mathbb{N} - g^{-1}(f^{-1}(f(g(\{c_1, c_2, \dots, c_{n-1}\}))))$$

How to think of this: Let $h = f \circ g$ be the composite function $h: \mathbb{N} \to B$. Then $g^{-1}(f^{-1}(f(g(\cdot)))) = h^{-1}(h(\cdot))$. So we are letting c_n be the smallest number with the property that $h(c_n) \notin \{h(c_1), h(c_2), \dots, h(c_{n-1})\}$.

Let $h = f \circ g$ be the composite. I claim

$h: \{c_1, c_2, c_3, \dots\} \to B$ is bijective.

Injective: By definition, $h(c_n)$ is distinct from $h(c_1), \dots, h(c_{n-1})$. So h is injective.

Surjective: Pick any $b \in B$. Then since g and f are surjective, there's some $c \in \mathbb{N}$, $h(c) = b$. Pick c to be the smallest such number. It's easily seen that $c = c_i$ for some i. ∎

Cor: If A and B are denumerable, so is $A \cup B$.

Pf: Let $f_A: \mathbb{N} \to A$, $f_B: \mathbb{N} \to B$ be bijective. Define

$$f: \mathbb{N} \times \{0,1\} \to A \cup B$$
$$f(n,0) = f_A(n)$$
$$f(n,1) = f_B(n)$$

This function is obviously surjective. Since $\mathbb{N} \times \{0,1\}$ is denumerable, so is $A \cup B$.

Ex: $\{n \in \mathbb{Z} : n \text{ is even or } 3 \mid n\} = \underbrace{\{n \in \mathbb{Z} : 2 \mid n\}}_{\text{denom}} \cup \underbrace{\{n \in \mathbb{Z} : 3 \mid n\}}_{\text{denom}}$